Equilibria of nonlinear distorted Brownian motions

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References:
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1. From McKean-Vlasov SDEs to nonlinear FP(K)Es and back

a) McKean-Vlasov SDEs $\longrightarrow$ nonlinear FPEs

Let $\mathcal{P}(\mathbb{R}^d)$ denote the set of all Borel probability measures on $\mathbb{R}^d$ and let

$$b = (b_1, \dots, b_d) \colon [0,\infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \longrightarrow \mathbb{R}^d,$$
$$\sigma = (\sigma_{ij})_{1 \le i,j \le d} \colon [0,\infty) \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \longrightarrow L(\mathbb{R}^d, \mathbb{R}^d)$$

be **measurable**. Consider the following McKean-Vlasov SDE

$$\mathrm{d}X(t) = b\big(t, X(t), \mathcal{L}_{X(t)}\big)\,\mathrm{d}t + \sigma\big(t, X(t), \mathcal{L}_{X(t)}\big)\,\mathrm{d}W(t), \qquad \text{(MVSDE)}$$

where $W(t)$, $t \ge 0$, is an $\mathbb{R}^d$-valued $(\mathcal{F}_t)$-Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and

$$\mathcal{L}_{X(t)} := \mathbb{P} \circ X(t)^{-1}, \quad t \ge 0,$$

are the time marginal laws of $X(t)$, $t \ge 0$, under $\mathbb{P}$.

By Itô's formula it is easy to find the nonlinear (!) Fokker–Planck equation (FPE for short) for the **time marginal laws** $\boldsymbol{\mathcal{L}_{X(t)} =: \mu_t}$, $\boldsymbol{t} \geq 0$, of the solution $X(t)$, $t \geq 0$, to (MVSDE). More precisely, for smooth $\varphi : \mathbb{R}^d \to \mathbb{R}$ with compact support we have for $t \geq 0$

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi(x)\mu_t(\mathrm{d}x) &= \int_\Omega \varphi(X(t)(\omega))\,\mathbb{P}(\mathrm{d}\omega) \\ &= \int_\Omega \varphi(X(0)(\omega))\,\mathbb{P}(\mathrm{d}\omega) + \int_\Omega \int_0^t L_{\mathcal{L}_{X(s)}}\varphi(X(s)(\omega))\,\mathrm{d}s\,\mathbb{P}(\mathrm{d}\omega) \\ &= \int_{\mathbb{R}^d} \varphi(x)\mu_0(\mathrm{d}x) + \int_0^t \int_{\mathbb{R}^d} L_{\mu_s}\varphi(s,x)\mu_s(\mathrm{d}x)\mathrm{d}s \qquad \text{(NLFPE)} \end{aligned}$$

where for $x \in \mathbb{R}^d$, $t \geq 0$, and $a_{ij} := (\sigma\sigma^T)_{ij}$, $1 \leq i,j \leq d$,

$$L_{\mu_t}\varphi(t,x) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(t,x,\mu_t)\frac{\partial^2}{\partial x_i \partial x_j}\varphi(x) + \sum_{i=1}^d b_i(t,x,\mu_t)\frac{\partial}{\partial x_i}\varphi(x).$$

References: **Huge!** E.g. Carmona/Delarue: Vol. I + II, Springer 2018

We can rewrite (NLFPE) in the sense of Schwartz distributions as follows:

$$\frac{\partial}{\partial t}\mu_t = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i \partial x_j}\big[a_{ij}(t,x,\mu_t)\mu_t\big] - \sum_{i=1}^{d}\frac{\partial}{\partial x_i}\big[b_i(t,x,\mu_t)\mu_t\big],$$

$\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ given,

or shortly

$$\partial_t \mu = \frac{1}{2}\partial_i\partial_j(a_{ij}(\mu)\mu) - \partial_i(b_i(\mu)\mu), \qquad \text{(“distributional solution”)}$$

$\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ given.

We refer to Chap. 10 in: Bogachev/Krylov/R./Shaposhnikov: AMS Monograph 2015.

b) Nonlinear FPEs $\longrightarrow$ McKean-Vlasov SDEs

Now let us go backwards, i.e. first solve (NLFPE) **and then construct a weak solution to** (MVSDE)**.**

Let a_{ij}, b_i, $1 \le i, j \le d$, be as in the previous section.

Assumption: There **exists** a solution $[0, \infty) \ni t \mapsto \boldsymbol{\mu}_t \in \mathcal{P}(\mathbb{R}^d)$ of (NLFPE) such that

(i) For all $T > 0$ and $1 \le i, j \le d$

- $a_{ij}, b_i \in L^1([0, T] \times U, \boldsymbol{\mu}_t \mathrm{d}t)$ for every ball $U \subset \mathbb{R}^d$,
- $\displaystyle\int_0^T \int_{\mathbb{R}^d} \frac{|a_{ij}(t, x, \boldsymbol{\mu}_t)| + |\langle x, b_i(t, x, \boldsymbol{\mu}_t)\rangle|}{1 + |x|^2} \, \boldsymbol{\mu}_t(\mathrm{d}x) \, \mathrm{d}t < \infty$

(ii) $[0, \infty) \ni t \mapsto \boldsymbol{\mu}_t$ is weakly continuous.

Now **fix** this solution $(\mu_t)_{t\geq 0}$.

Theorem I ([Barbu/R.: SIAM Journal of Math. Analysis 2018 and Ann. Probab. 2020])

There exists a d-dimensional $(\mathcal{F}_t)$*-Brownian motion* $W(t)$, $t \geq 0$, *on a stochastic basis* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *and a continuous* $(\mathcal{F}_t)$*-progressively measurable map* $X : [0, \infty) \times \Omega \to \mathbb{R}^d$ *satisfying the following McKean-Vlasov SDE*

$$\mathrm{d}X(t) = b\big(t, X(t), \boldsymbol{\mu}_t\big)\,\mathrm{d}t + \sigma\big(t, X(t), \boldsymbol{\mu}_t\big)\,\mathrm{d}W(t),$$

where $\sigma = ((a_{ij})_{1\leq i,j\leq d})^{\frac{1}{2}}$, *such that we have, for the marginals,*

$$\mathcal{L}_{X(t)} = \mathbb{P} \circ X(t)^{-1} = \boldsymbol{\mu}_t, \quad t \geq 0. \qquad (\text{“probabilistic representation”}) \qquad \text{(PR)}$$

Remark

b, σ **only measurable** in measure variable !

Proof. Let $(\mu_t)_{t\geq 0}$ be as in **Assumption**. Then by [Bogachev/R./Shaposhnikov: JDDE 2020], which is a recent regeneralization of a beautiful result in [Trevisan: EJP 2016], there exists a probability measure P on $C([0,T];\mathbb{R}^d)$ equipped with its Borel σ-algebra and its natural filtration generated by the evaluation maps π_t, $t \in [0,T]$, defined by

$$\pi_t(w) := w(t),\ \ w \in C([0,T],\mathbb{R}^d),$$

solving the martingale problem for the **linear** Kolmogorov operator (with $\boldsymbol{\mu} = (\boldsymbol{\mu}_t)_{t\geq 0}$ as above **fixed**)

$$L_u := \tfrac{1}{2} a_{ij}(\boldsymbol{\mu})\partial_i\partial_j + b_i(\boldsymbol{\mu})\partial_i$$

with marginals

$$P \circ \pi_t^{-1} = \boldsymbol{\mu}_t, \quad t \geq 0.$$

Then, the assertion follows by a standard result (see e.g. [Stroock: LMS Text 1987]).

c) The Nemytskii – case

The dependence of a_{ij} and b_i, $1 \leq i,j \leq d$, on the measure $\boldsymbol{\mu}_t(\mathrm{d}x)$ can be arbitrary (as long as it is measurable). In Section 2 we shall, however, consider examples of the following type: we look for a solution $(\boldsymbol{\mu}_t)_{t\geq 0}$ to (NLFPE), which is absolutely continuous, i.e.

$$\boldsymbol{\mu}_t(\mathrm{d}x) = \boldsymbol{u}(t,x)\,\mathrm{d}x, \quad t \geq 0,$$

($\mathrm{d}x =$ Lebesgue measure on $\mathbb{R}^d$) and a_{ij}, b_i are of Nemytskii–type, i.e. for $t \geq 0$, $x \in \mathbb{R}^d$,

$$\begin{aligned} a_{ij}(t,x,\boldsymbol{u}(t,\cdot)\,\mathrm{d}x) &= \overline{a_{ij}}(t,x,\boldsymbol{u}(t,x)), \\ b_i(t,x,\boldsymbol{u}(t,\cdot)\,\mathrm{d}x) &= \overline{b_i}(t,x,\boldsymbol{u}(t,x)), \end{aligned} \qquad \text{"Nemytskii–type"}$$

where

$$\begin{aligned} \overline{a_{ij}}&\colon [0,\infty) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}, \\ \overline{b_i}&\colon [0,\infty) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \end{aligned}$$

are measurable functions.

Remark

No continuity in the measure variable !

Then the NLFPE is

$$\frac{\partial}{\partial t}\boldsymbol{u}(t,x)=\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i\partial x_j}\big[\overline{a_{ij}}(t,x,\boldsymbol{u}(t,x))\boldsymbol{u}(t,x)\big]-\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\big[\,\overline{b_i}(t,x,\boldsymbol{u}(t,x))\boldsymbol{u}(t,x)\big],$$

$$u(0,\cdot)\text{ a given probability density on }\mathbb{R}^d,$$

and for $\sigma\sigma^T=(\overline{a_{ij}})_{1\le i,j\le d}$ the corresponding McKean–Vlasov SDE is

$$\begin{aligned}\mathrm{d}X(t)&=\overline{b}(t,X(t),\boldsymbol{u}(t,X(t)))\mathrm{d}t+\sigma(t,X(t),\boldsymbol{u}(t,X(t)))\mathrm{d}W(t),\\ \mathcal{L}_{X(t)}(\mathrm{d}x)&=\boldsymbol{u}(t,x)\mathrm{d}x,\quad t\ge 0.\end{aligned}$$

Note: Theorem I above still applies, if **Assumption** holds. So, the **task** is to solve the above NLFPE and check **Assumption** for its solution $\boldsymbol{u}(t,x)\mathrm{d}x,\ t\in[0,T]$. Then vision from [McKean: PNAS 1966] realized!

2. Perturbed porous media equation and nonlinear distorted Brownian motion

Ref.: [Barbu/R.: arXiv:1904.08291v7, IUMJ 2021+]
In this section we look at the following special Nemytskii–type NLFPKE

$$\begin{aligned} u_t - \tfrac{1}{2}\Delta\beta(u) + \mathrm{div}(Db(u)u) = 0 \text{ in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x),\ x\in\mathbb{R}^d, \end{aligned} \tag{pPME}$$

where $d\in\mathbb{N}$ and $\beta:\mathbb{R}\to\mathbb{R}$, $D:\mathbb{R}^d\to\mathbb{R}^d$ and $b:\mathbb{R}\to\mathbb{R}$, such that

(i) $\beta\in C^1(\mathbb{R}),\ \beta(0)=0,\ \gamma\le\beta'(r)\le\gamma_1,\ \forall r\in\mathbb{R}$, for $0<\gamma<\gamma_1<\infty$.

(ii) $b\in C_b(\mathbb{R})\cap C^1(\mathbb{R})$.

(iii) $D\in L^\infty(\mathbb{R}^d;\mathbb{R}^d)\cap W^{1,1}_{loc}(\mathbb{R}^d;\mathbb{R}^d)$, *div* $D\in(L^\infty(\mathbb{R}^d)+L^1(\mathbb{R}^d))\cap(L^\infty(\mathbb{R}^d)+L^2(\mathbb{R}^d))$.

(iv) $D=-\nabla\Phi$, *where* $\Phi\in W^{2,1}_{loc}(\mathbb{R}^d),\ \Phi\ge1,\ \lim_{|x|_d\to\infty}\Phi(x)=+\infty$ *and there exists* $m\in(0,\infty)$ *such that* $\Phi^{-m}\in L^1(\mathbb{R}^d)$

(hence $\overline{a_{ij}}(t,x,r) := \frac{\beta(r)}{r}\delta_{ij}, \overline{b_i}(t,x,r) := b(r)D(x), r\in\mathbb{R}$).

Its corresponding Kolmogorov operator is

$$L_{u(t,x)} = \frac{1}{2}\frac{\beta(u(t,x))}{u(t,x)}\Delta - b(u(t,x))\langle\nabla\Phi, \nabla\cdot\rangle$$

Generator of distorted Brownian motion if $\beta = id$ and $b = const.$!

and the corresponding DD (= McKean–Vlasov) SDE

$$\mathrm{d}X(t) = -b(\mathcal{L}_{X(t)}(X(t)))\nabla\Phi(X(t))\mathrm{d}t + \sqrt{\frac{\beta(\mathcal{L}_{X(t)}(X(t)))}{\mathcal{L}_{X(t)}(X(t))}}\mathrm{d}W(t),$$

$$\mathcal{L}_{X(t)}(x) := \frac{\mathrm{d}\mathcal{L}_{X(t)}}{\mathrm{d}x}(x) = u(t,x), \quad t \geq 0, \qquad (NLDBM)$$

"nonlinear distorted Brownian motion"

$$\mathcal{L}_{X(t)}(dx) = \mathbb{P}\circ X(t)^{-1}(dx), \quad t \geq 0.$$

Remark

We shall see that by Theorem I above and Theorem II below, the above DDSDE has a weak solution, so **"nonlinear distorted Brownian motion"** exists.

Remark

A typical example for Φ *as in* (iv) above is

$$\Phi(x) = C(1+|x|^2)^{\alpha},\ x \in \mathbb{R}^d,$$

with $\alpha \in \left(0, \frac{1}{2}\right]$.

Now let us solve (pPME).
Consider the operator $A : D(A) \subset L^1 \to L^1$, defined by

$$\begin{aligned} Au &:= -\tfrac{1}{2}\Delta\beta(u) + \operatorname{div}(Db(u)u),\ \forall u \in D(A), \\ D(A) &:= \{u \in L^1;\ -\Delta\beta(u) + \operatorname{div}(Db(u)u) \in L^1\}, \end{aligned}$$

in $L^1 := L^1(\mathbb{R}^d)$. Here, the differential operators Δ and div are taken in the sense of Schwartz distributions, i.e., in $\mathcal{D}'(\mathbb{R}^d)$. Obviously, the operator $(A, D(A))$ is closed on L^1. Denote by $\overline{D(A)}$ the closure of $D(A)$ in L^1.

Proposition I

Assume that Hypotheses (i)–(iv) *hold. Then, the operator A is m-accretive, that is,*

$$R(I+\lambda A)=L^1,\ \forall\lambda>0,$$

$$|(I+\lambda A)^{-1}u-(I+\lambda A)^{-1}v|_1\le|u-v|_1,\ \forall\lambda>0,\ u,v\in L^1.$$

Furthermore,

$$\overline{D(A)}=L^1,$$

where "$\overline{\quad}$" denotes the closure in L^1. Moreover, there exists $\lambda_0>0$ such that, for all $\lambda\in(0,\lambda_0)$,

$$\int_{\mathbb{R}^d}(I+\lambda A)^{-1}u_0dx=\int_{\mathbb{R}^d}u_0(x)dx,\ \forall u_0\in L^1,$$

$$(I+\lambda A)^{-1}u_0\ge 0,\quad a.e.\ in\ \mathbb{R}^d\ if\ u_0\ge 0,\ a.e.\ in\ \mathbb{R}^d.$$

Now we can rewrite (pPME) as the Cauchy problem associated with A, that is,

$$\begin{aligned} &\frac{du}{dt} + Au = 0, \ \ t \geq 0, \\ &u(0) = u_0. \end{aligned} \tag{CP}$$

Since A is m-accretive, we have by the Crandall & Liggett theorem the following existence result for (CP).

Theorem II ([Barbu/R.: arXiv:1904.08291v7, IUMJ 2021+])

Under Hypotheses (i)–(iv)*, for every* $u_0 \in L^1$*, and* $t \geq 0$*, the following limit exists*

$$u(t) = \lim_{n\to\infty} \left(I + \frac{t}{n} A\right)^{-n} u_0 =: S(t)u_0 \, (= "e^{-tA}" u_0),$$

uniformly on bounded intervals of $[0, \infty)$ *in* L^1 *and is called mild solution to* (CP)*. Furthermore,*

$$\int_{\mathbb{R}^d} u(t,x)dx = \int_{\mathbb{R}^d} u_0(x)dx, \ \forall t \geq 0,$$

$$u(t,x) \geq 0, \ \textit{a.e. on } (0,\infty) \times \mathbb{R}^d \ \textit{if } u_0 \geq 0, \ \textit{a.e. in } \mathbb{R}^d.$$

In particular, for each $t \geq 0$, $u(t, \cdot)$ *is a probability density if so is* u_0*.*

Theorem II (continued)

Furthermore, the map $t \to S(t)u_0$ is a continuous semigroup of (nonlinear) contractions on L^1, that is,

$$S(t+s)u_0 = S(t)S(s)u_0,\ \forall t,s \geq 0,\ u_0 \in L^1,$$

$$\lim_{t\to 0} S(t)u_0 = u_0 \text{ in } L^1,$$

$$|S(t)u_0 - S(t)\bar{u}_0|_1 \leq |u_0 - \bar{u}_0|_1,\ \forall t \geq 0,\ u_0, \bar{u}_0 \in L^1.$$

In particular, $u(t,\cdot) = S(t)u_0,\ t \geq 0$, is a distributional solution to (pPME) and, if $u_0 \in L^1 \cap L^\infty$, it is the unique solution in $L^1([0,T]\times\mathbb{R}^d) \cap L^\infty([0,T]\times\mathbb{R}^d)$.

Theorem I + II $\Longrightarrow$

Theorem III

There exists a probabilistically weak solution to (NLDBM).
Furthermore, by [Barbu/R.: arXiv: 1909.04464v2, SPDE 2021+] for $u_0 \in L^1 \cap L^\infty$ it is unique, among all with time marginals in $L^\infty([0,T]\times\mathbb{R}^d)$

3. Asymptotic behaviour and unique stationary solution: The H–Theorem

Additionally to (i)-(iv), assume

(v) $b(r) \geq b_0 \in (0, \infty) \quad \forall r \geq 0.$

(vi) $\gamma_1 \Delta\Phi - b_0 |\nabla\Phi|^2 \leq 0$ **("balance condition")**

Consider the following subspace of L^1

$$\mathcal{M} = \left\{ u \in L^1; \int_{\mathbb{R}^d} \Phi(x)|u(x)|dx < \infty \right\} \text{ with norm } \|u\| := \int_{\mathbb{R}^d} \Phi(x)|u(x)|dx.$$

Proposition II (one key point)

$$\|S(t)u_0\| \leq \|u_0\| \;\forall u_0 \in \mathcal{M}^+.$$

Define $\eta \in C([0,\infty)) \cap C^2((0,\infty))$ by

$$\eta(r) := \int_0^r d\tau \int_1^\tau \frac{\beta'(s)}{sb(s)}\, ds,\ \ r \in (0,\infty),$$

Note that $\eta(r) = r(log\ \ r - 1)$, if $\beta(s) = s, b(s) = 1\ \forall\ s \in \mathbb{R}$.

Define

$$V : D(V) = \{u \in \mathcal{M};\ \ u \geq 0,\ \ \mathrm{u\ log\ u} \in L^1\} \to \mathbb{R}$$

by

$$V(u) := \underbrace{\int_{\mathbb{R}^d} \eta(u(x))dx}_{= \,-\text{Entropy}} + \underbrace{\int_{\mathbb{R}^d} \Phi(x)u(x)dx}_{= \text{ Energy}} = -S[u] + E[u], u \in M, u \geq 0.$$

Define for $u_0 \in D(V)$

$$\omega(u_0) := \{L^1 - \lim_{t_n \to \infty} S(t_n)u_0 : \{t_n\} \to \infty\}\ (\text{"}\omega - \text{limit set of } u_0.\text{"})$$

Theorem IV ("H-Theorem", [Barbu/R.: arXiv:1904.08291v7, IUMJ 2021+])

Assume that Hypotheses (i)–(vi) *above hold. Then the function V is a* **Lyapunov function** for $S(t)$, $t \geq 0$, that is, $\forall u_0 \in D(V)(:= \mathcal{M}^+ \cap L \log L)$ and $0 \leq s \leq t < \infty$

$$S(t)u_0 \in D(V) \text{ and } V(S(t)u_0) \leq V(S(s)u_0).$$

Moreover, for all $u_0 \in D(V) \setminus \{0\}$ there exists $u_\infty \in D(V) \cap L^\infty$ such that

$$\omega(u_0) = \{u_\infty\},$$

$u_\infty > 0$ a.e., $|u_\infty|_1 = |u_0|_1$, and it is given by

$$u_\infty = g^{-1}(-\Phi + \mu) \text{ with } g(r) := \int_1^r \frac{\beta'(s)}{sb(s)} ds, r \in (0, \infty),$$

where μ is the unique number in $\mathbb{R}$ such that

$$\int_{\mathbb{R}^d} g^{-1}(-\Phi(x) + \mu) dx = \int_{\mathbb{R}^d} u_0 dx.$$

Theorem IV ("H-Theorem" continued)

In particular, for all $u_0 \in D(V)$ *the (one point) set* $\omega(u_0) = \{u_\infty\}$ *only depends on* $|u_0|_1$ *Furthermore,* u_∞ *is the unique element in* $D(V)$ *with given* L^1*-norm such that* $S(t)u_\infty = u_\infty$ *for all* $t \geq 0$ *and* $u_\infty dx$ *is the unique stationary probability (distributional) solution with density in* L^∞ *of equation* (pPME)*. Consequently,* $u_\infty dx$ *is the unique invariant measure with density in* L^∞ *for the* **"nonlinear distorted Brownian motion"**.

Proof. Important ingredient: modification of a general technique in [Pazy: J. d'Analyse Math. 1981].

Remark u_∞ is stationary solution of (pPME), according to Theorem IV.

Heuristical proof.

Set $u := u_\infty$.

$$div(-\nabla\beta(u) - \nabla\Phi b(u)u) = -\Delta\beta(u) - div(\nabla\Phi b(u)u) = 0$$

$$\Longleftarrow \underbrace{\frac{\beta'(u)}{ub(u)}\nabla u}_{= \nabla\left(\int_1^u \frac{\beta'(s)}{sb(s)}ds\right) = \nabla g(u)} = -\nabla\Phi$$

$$\Longleftrightarrow g(u) = -\Phi + \mu \text{ for some } \mu \text{ in } \mathbb{R}$$

$$\Longleftrightarrow u = g^{-1}(-\Phi + \mu).$$

4. Degenerate nonlinear distorted Brownian motion

As before we look at the following special Nemytskii–type NLFPKE

$$\begin{aligned} u_t - \tfrac{1}{2}\Delta\beta(u) + \operatorname{div}(Db(u)u) = 0 \text{ in } (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x),\ x\in\mathbb{R}^d, \end{aligned} \tag{pPME}$$

where $d\in\mathbb{N}$ and $\beta:\mathbb{R}\to\mathbb{R}$, $D:\mathbb{R}^d\to\mathbb{R}^d$ and $b:\mathbb{R}\to\mathbb{R}$, Except for (i) our Hypotheses are as before:

(i)' $\beta\in C^1(\mathbb{R}),\ \beta'(r)>0,\ \forall\, r\in\mathbb{R}\setminus\{0\},\ \beta(0)=0$, and $\mu_1\min\{|r|^\nu,|r|\}\le|\beta(r)|\le\mu_2|r|\ \forall\, r\in\mathbb{R}$, for $\mu_1,\mu_2>0$ and $\nu>\frac{d-1}{d},\ d\ge 3$.

(ii) $b\in C_b(\mathbb{R})\cap C^1(\mathbb{R})$.

(iii) $D\in L^\infty(\mathbb{R}^d;\mathbb{R}^d)\cap W^{1,1}_{loc}(\mathbb{R}^d;\mathbb{R}^d)$, *div* $D\in(L^\infty(\mathbb{R}^d)+L^1(\mathbb{R}^d))\cap(L^\infty(\mathbb{R}^d)+L^2(\mathbb{R}^d))$.

(iv) $D=-\nabla\Phi$, *where* $\Phi\in W^{2,1}_{loc}(\mathbb{R}^d)$, $\Phi\ge 1$, $\lim\limits_{|x|_d\to\infty}\Phi(x)=+\infty$ *and there exists* $m\in(0,\infty)$ *such that* $\Phi^{-m}\in L^1(\mathbb{R}^d)$

(v) $b(v)\ge b_0\in(0,\infty)$

(vi) ("balance condition") $\mu_2\Delta\Phi - b_0|\nabla\Phi|^2\le 0$ a.e. on $\mathbb{R}^d$

Then the existence (not yet uniqueness) result from Theorems II and III above still hold. What about the (H-) Theorem IV?

Lyapunov function is still the same, namely:

$$V(u) := \underbrace{\int_{\mathbb{R}^d} \eta(u(x))dx}_{= -\text{Entropy}} + \underbrace{\int_{\mathbb{R}^d} \Phi(x)u(x)dx}_{= \text{Energy}}, \ u \in \mathcal{M}^+ \cap \text{ L log L},$$

where as before

$$\eta(r) := \int_0^r d\tau \underbrace{\int_1^\tau \frac{\beta'(s)}{sb(s)} ds}_{=: g(\tau)}, \ r \in (0, \infty),$$

For R $\in (0,\infty)$ let

$$\mathcal{M}_R := \{u \in \mathcal{M} | \|u\| \leq R\}$$

and let

$$\mathcal{P} := \{u \in L^1 | u \geq 0, \int_{\mathbb{R}^d} u \, dx = 1\}.$$

Theorem V ("Existence of a stationary solution / invariant measure")

[Barbu/R.: arXiv:2105.02328]
Suppose Hypotheses (i)', (ii)-(vi) hold and that, additionally,

$$\lim_{r\to\infty} g(r) = \infty, \text{ if } \nu \in (\tfrac{d-1}{d}, 1]$$

and $\lim_{r\to 0} g(r) = -\infty, \text{ if } \nu \in (1,\infty).$

Then $\exists u_\infty \in \mathcal{M} \cap \mathcal{P} \cap D(A) \cap L^\infty$ *such that*

$$S(t)u_\infty = u_\infty \ \forall t \geq 0.$$

4. Degenerate nonlinear distorted Brownian motion

Theorem VI ([Barbu/R.: arXiv:2105.02328])

Suppose Hypotheses (i)', (ii)-(vi) hold. Let $R \in (0, \infty)$ *and let* $u_0 \in \mathcal{M}_R \cap \mathcal{P} \cap \overline{D(A)}^{L^1}$*. Then* $\omega(u_0) \subset \mathcal{M}_R \cap \mathcal{P} \cap \overline{D(A)}^{L^1}$ *is nonempty and for all* $t \geq 0$*,* $\omega(u_0)$ ***is compact in*** $\boldsymbol{L^1}$**,** $\omega(u_0) = \overline{\{S(t)u_0 \mid t \geq 0\}}^{L^1}$, and invariant under $S(t)$. Moreover, $S(t)$ is, for every $t \geq 0$, an isometry on $\omega(u_0)$ and it is a homeomorphism from $\omega(u_0)$ onto itself for each $t \geq 0$. If $a \in \mathcal{M}_R \cap \mathcal{P} \cap \overline{D(A)}^{L^1}$ is such that

$$S(t)a = a, \quad \forall\, t \geq 0, \tag{4.1}$$

then $\omega(u_0) \subset \{y \in \mathcal{M}_R \cap \mathcal{P} \cap \overline{D(A)}^{L^1};\ |y - a|_1 = r\}$, for some $0 \leq r \leq |u_0 - a|_1$.

Proof. Important ingredient: a modification of a general result in [Dafermos/Slemrod: JFA 1973]. Hard part: check the assumption, that $\omega(u_0)$ is compact in L^1.